New solvable singular potentials

2001 J. Phys. A: Math. Gen. 34 4129

(http://iopscience.iop.org/0305-4470/34/19/311)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 4129–4142 www.iop.org/Journals/ja PII: S0305-4470(01)22138-6

New solvable singular potentials

R Dutt[1], **A Gangopadhyaya**[2], **C Rasinariu**[3] **and U Sukhatme**[4]

[1] Department of Physics, Visva Bharati University, Santiniketan 731235, India
[2] Department of Physics, Loyola University Chicago, Chicago, IL 60626, USA
[3] Department of Science and Mathematics, Columbia College Chicago, Chicago, IL 60605, USA
[4] Department of Physics (m/c 273), University of Illinois at Chicago, Chicago, IL 60607, USA

E-mail: rdutt@cal2.vsnl.net.in, agangop@luc.edu, asim@uic.edu, crasinariu@popmail.colum.edu, costel@uic.edu and sukhatme@uic.edu

Received 16 February 2001, in final form 9 March 2001

Abstract

We obtain three new solvable, real, shape-invariant potentials starting from the harmonic oscillator, Pöschl–Teller I and Pöschl–Teller II potentials on the half-axis and extending their domain to the full line, while taking special care to regularize the inverse-square singularity at the origin. The regularization procedure gives rise to a delta-function behaviour at the origin. Our new systems possess underlying nonlinear potential algebras, which can also be used to determine their spectra analytically.

PACS numbers: 0365, 1130P

1. Introduction

In recent years, several authors have investigated the eigenstates of complex potentials [1], especially those with PT symmetry and real spectra. In particular, potentials obtained by replacing the real coordinate x by a complex variable $x + ic$ in a class of exactly solvable shape-invariant potentials has been considered by Znojil [2]. For these problems, which are defined on the whole real line, an extension to the complex domain was made in order to avoid an inverse-square singularity at the origin. The price one pays for taming the singularity is to deal with a complex potential. However, it was argued that owing to the PT-symmetric nature of the potential, the eigenvalues were still real. As an explicit example, it was shown [2] that a new exactly solvable complex harmonic-oscillator-like potential with two shifted sets of equally spaced energy levels could be generated. The same technique was also applied to explicitly get the eigenstates of complex Pöschl–Teller-I- and Pöschl–Teller-II-like potentials [3].

One of the purposes of this paper is to show that potentials with an inverse-square singularity at the origin do not necessarily call for moving into the complex domain. In fact, we can obtain the spectra of [2, 3] simply by judicious application of the formalism of supersymmetric quantum mechanics (SUSYQM). Specifically, the spectrum described in [2] for the harmonic-oscillator-like potential is identical to that previously found by us [4], where the discussion focused on real potentials with two sets of equally spaced eigenvalues. In this paper, we extend previous results and also find new real but singular potentials corresponding to the Pöschl–Teller I and Pöschl–Teller II potentials. Our potentials are shape invariant [5], and consequently their exact spectra can be obtained by standard algebraic procedures followed in SUSYQM. We also establish that these singular potentials possess an interesting underlying nonlinear potential algebra [6–10]. Explicit representations of the generators are given, which provide an alternative algebraic approach to determine the spectrum.

For completeness, we provide in section 2 a brief review of SUSYQM [11,12]. In section 3, we present our framework for generating new shape-invariant potentials starting from well known solvable problems with an inverse-square singularity at the origin. We show that if the coefficient of this singularity is restricted within a narrow range, one can enlarge the domain of the potential to the negative real axis while maintaining unbroken supersymmetry and shape invariance. Working with an explicit example of a harmonic oscillator with an inverse-square singularity and using the formalism of SUSYQM, we show that such an extension necessitates the introduction of a δ-function at the origin [4] which eventually yields a non-equidistant spectrum for the system. We show that similar extensions can be made for Pöschl–Teller I and Pöschl–Teller II potentials as well and thus generate new shape-invariant potentials. We explicitly derive their eigenenergies and eigenfunctions. For these potentials, it is important to note that the eigenenergies depend on two parameters, *both* of which are transformed in the shape invariance condition, in contrast to previous work on shape invariance. In section 4, we study the potential algebra underlying these systems and generate their spectrum by algebraic means.

2. Supersymmetric quantum mechanics

In SUSYQM [12], taking $\hbar = 2m = 1$, the partner potentials $V_\pm(x, a_0)$ are related to the superpotential $W(x, a_0)$ by

$$V_\pm(x, a_0) = W^2(x, a_0) \pm W'(x, a_0) \tag{1}$$

where a_0 is a set of parameters. It is assumed that the superpotential $W(x)$ is continuous and differentiable. The corresponding Hamiltonians $H_\pm$ have a factorized form

$$H_- = \mathcal{A}^\dagger \mathcal{A} \qquad H_+ = \mathcal{A}\mathcal{A}^\dagger \qquad \mathcal{A} = \frac{\mathrm{d}}{\mathrm{d}x} + W(x) \qquad \mathcal{A}^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x). \tag{2}$$

We consider the case of unbroken supersymmetry and take $\psi_0 \sim \exp\left(-\int^x W(y)\,\mathrm{d}y\right)$ to be normalizable. This is clearly the nodeless zero-energy ground-state wavefunction for H_-, since $\mathcal{A}\psi_0 = 0$.

The Hamiltonians H_+ and H_- have exactly the same eigenvalues except that H_- has an additional zero-energy eigenstate. More specifically, the eigenstates of H_+ and H_- are related by

$$E_0^{(-)} = 0 \qquad E_{n-1}^{(+)} = E_n^{(-)} \qquad \psi_{n-1}^{(+)} \propto \mathcal{A}\,\psi_n^{(-)} \qquad \mathcal{A}^\dagger\,\psi_n^{(+)} \propto \psi_{n+1}^{(-)} \qquad n = 1, 2, \ldots. \tag{3}$$

Supersymmetric partner potentials are called shape invariant if they both have the same x-dependence up to a change of parameters $a_1 = f(a_0)$ and an additive constant which we denote by $R(a_0)$ [13, 14]. Often, it is convenient to write this constant in the form of $g(a_1) - g(a_0)$. The shape invariance condition is

$$V_+(x, a_0) = V_-(x, a_1) + R(a_0) = V_-(x, a_1) + g(a_1) - g(a_0). \tag{4}$$

The property of shape invariance permits an immediate analytic determination of energy eigenvalues [5, 13], and eigenfunctions [14]. If the change of parameters $a_0 \to a_1$ does not break supersymmetry, $H_-(x, a_1)$ also has a zero-energy ground state and the corresponding eigenfunction is given by $\psi_0^{(-)}(x, a_1) \propto \exp\left(-\int_{x_0}^{x} W(y, a_1)\,\mathrm{d}y\right)$. Now using equations (3), (4) we have

$$E_1^{(-)} = R(a_0) \qquad \psi_1^{(-)}(x, a_0) = \mathcal{A}^\dagger(x, a_0)\,\psi_0^{(+)}(x, a_0) = \mathcal{A}^\dagger(x, a_0)\,\psi_0^{(-)}(x, a_1). \tag{5}$$

Thus for an unbroken supersymmetry, the eigenstates of the potential $V_-(x)$ are

$$\begin{aligned} &E_0^{(-)} = 0 \qquad E_n^{(-)} = \sum_{k=0}^{n-1} R(a_k) = \sum_{k=0}^{n-1} [g(a_{k+1}) - g(a_k)] = g(a_n) - g(a_0) \\ &\psi_0^{(-)} \propto \mathrm{e}^{-\int_{x_0}^{x} W(y,a_0)\,\mathrm{d}y} \\ &\psi_n^{(-)}(x, a_0) = \left[-\frac{\mathrm{d}}{\mathrm{d}x} + W(x, a_0)\right] \psi_{n-1}^{(-)}(x, a_1) \qquad (n = 1, 2, 3, \ldots). \end{aligned} \tag{6}$$

These formulae are valid provided the change of parameters $a_1 = f(a_0)$ maintains unbroken supersymmetry. In previous work on shape-invariant potentials, changes of parameters corresponding to translation $a_1 = a_0 + \beta$ [13] and scaling $a_1 = qa_0$ with $0 < q \leqslant 1$ [15] have been discussed. However, a reflection change of parameters $a_1 = -a_0$, even if it maintained shape invariance, was not acceptable since it could not maintain unbroken supersymmetry for the hierarchy of potentials built on H_-.

3. New singular shape-invariant potentials

The methodology of this paper for obtaining new shape-invariant potentials is as follows. One begins with a known shape-invariant potential, defined for $x \geqslant 0$, which has an inverse-square singularity λ/x^2 at the origin. This potential is fully solvable, with eigenfunctions which vanish at the origin. One now considers extending the domain to also include the region $x < 0$. This extension is possible only if $-\frac{1}{4} < \lambda < \frac{3}{4}$. If the strength of the singular term is restricted to be in this limited domain, the singularity is called 'soft', and the potential is said to be 'transitional' [16]. We shall show explicitly how a new change of parameters corresponding to the reflection $a_1 = -a_0$ is now admissible, since it maintains both shape invariance and unbroken supersymmetry, while still keeping the partner potentials in the soft singularity domain. We can then obtain eigenspectra using the shape invariance formalism. As explicit examples, we present detailed analyses for the harmonic oscillator, Pöschl–Teller I and Pöschl–Teller II potentials.

3.1. New shape-invariant potential obtained from the harmonic oscillator potential

Consider a particle constrained to move in a three-dimensional harmonic oscillator potential

$$V_-(x, l, \omega) = \frac{1}{4}\omega^2 x^2 + \frac{l(l+1)}{x^2} + \left(l - \frac{1}{2}\right)\omega \qquad (0 < x < \infty). \tag{7}$$

This potential is generated from the superpotential

$$W(x, l, \omega) = \frac{1}{2}\omega x + \frac{l}{x} \qquad l < 0. \tag{8}$$

The supersymmetric partner potential is

$$V_+(x, l, \omega) = \frac{1}{4}\omega^2 x^2 + \frac{l(l-1)}{x^2} + \left(l + \frac{1}{2}\right)\omega. \tag{9}$$

These two partner potentials are shape invariant since $V_+(x, l, \omega)$ can be written as $V_-(x, l - 1, \omega) + R(l, \omega)$. Here, the remainder $R(l, \omega) = 2\omega$ is independent of the parameter l. This

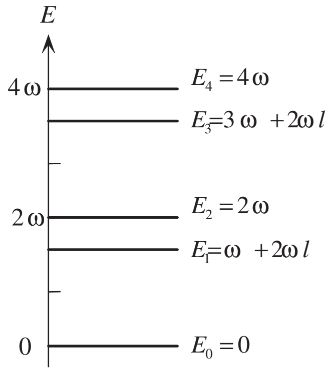

Figure 1. Energy eigenvalues corresponding to equation (10).

yields an equidistant spectrum $E_n = 2n\omega$ for the harmonic oscillator. However, there is another change of parameters that also maintains shape invariance between these two partner potentials, namely

$$V_+(x, l, \omega) = V_-(x, -l, \omega) + R'(l, \omega)$$

$R'(l, \omega) = (2l + 1)\omega$.

However, it is important to point out that since we are at present constrained to be on the half-axis $x > 0$, this second change of parameters, $(l, \omega) \longrightarrow (-l, \omega)$ is not acceptable for $l < 0$. Neither of the two zero-energy solutions $\psi_0^{(\pm)}(x, -l, \omega) \propto \exp\left(\pm \int^x W(x, -l, \omega)\, \mathrm{d}x\right)$ is normalizable and hence supersymmetry is spontaneously broken. As shown in section 2, the solvability of shape-invariant systems crucially depends upon superpotentials retaining unbroken supersymmetry when parameters are transformed, that is, it is essential that $V_-(x, a_1)$ be a potential with unbroken supersymmetry.

Let us now consider the same superpotential with an extension of the domain to the entire real axis. The asymptotic values of the superpotential are given by the $\frac{1}{2}\omega x$ term at $x \to \pm\infty$, which is independent of the parameter l. Thus the asymptotic behaviour of the ground-state wavefunction is dictated by the ωx-term and is not affected by flipping of the value of l in the $\frac{l}{x}$-term of the superpotential. Thus, in contrast with the half-axis case, supersymmetry now remains unbroken even with the change of parameters $(l, \omega) \longrightarrow (-l, \omega)$, and hence this transformation is allowed to generate new shape-invariant potentials with richer spectra. This leads to

$$E_n = n\omega + 2l\omega P_n \qquad P_n \equiv [1 - (-1)^n]/2. \tag{10}$$

This new shape invariance yields a new set of eigenenergies superimposed on the old equidistant spectrum and shown in figure 1.

We now focus on the region near $x = 0$. In SUSYQM, it is important that the superpotential $W(x, a_0)$ be a continuous and differentiable function. In our example, the above requirement is satisfied everywhere except at the point $x = 0$, where the superpotential of equation (8) has an infinite discontinuity. Such a discontinuity is not acceptable, and needs regularization. Consider a regularized, continuous superpotential $\tilde{W}(x, a_0, \epsilon)$ which reduces to $W(x, a_0)$ in the limit $\epsilon \to 0$. One such choice is

$$\tilde{W}(x, a_0, \epsilon) = W(x, a_0)\ f(x, \epsilon) \tag{11}$$

where

$$f(x, \epsilon) = \tanh^2 \frac{x}{\epsilon}. \tag{12}$$

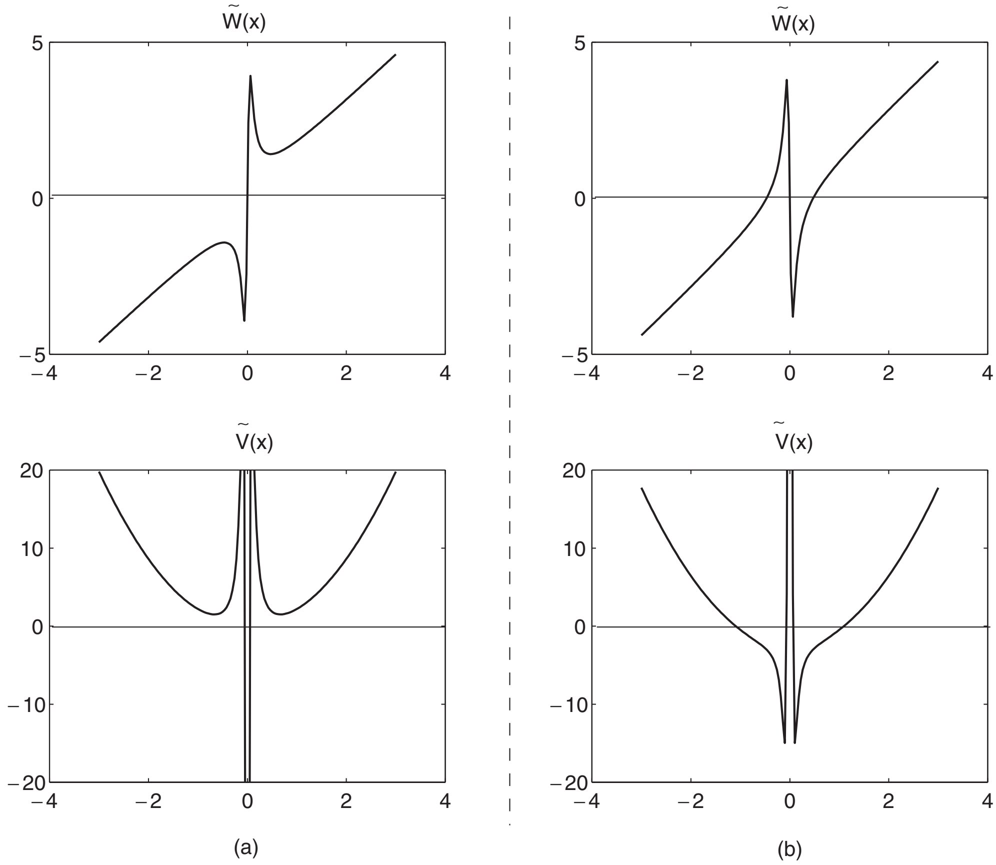

Figure 2. The superpotential $\tilde{W}$ of equation (11) and the corresponding potential $\tilde{V}_-$ of equation (13) for the two cases (*a*) $l \geqslant 0$ and (*b*) $l \leqslant 0$.

The moderating factor f provides a smooth interpolation through the discontinuity, since it is unity everywhere except in a small region of order ϵ around $x = 0$. In this region, $\tilde{W}(x, a_0, \epsilon)$ is linear with a slope l/ϵ^2. The potential[5] $\tilde{V}_-(x, a_0, \epsilon)$ corresponding to the superpotential $\tilde{W}(x, a_0, \epsilon)$ is

$$\tilde{V}_-(x, a_0, \epsilon) = \tilde{W}^2(x, a_0, \epsilon) - \tilde{W}'(x, a_0, \epsilon). \tag{13}$$

In the limit $\epsilon \to 0$, $\tilde{V}_-(x, a_0, \epsilon)$ reduces to

$$\tilde{V}_-(x, a_0) = V_-(x, a_0) - 4\, W(x, a_0)\, \frac{x}{|x|}\, \delta(x) \tag{14}$$

where we have used $\lim_{\epsilon\to 0} \frac{1}{2\epsilon} \operatorname{sech}^2 \frac{x}{\epsilon} = \delta(x)$ and $\lim_{\epsilon \to 0} \tanh \frac{x}{\epsilon} = \frac{x}{|x|}$. Thus we see that the potential $\tilde{V}_-(x, a_0)$ has an additional singularity at the origin over $V_-(x, a_0)$ given by $\Omega(x) \equiv -4l\, \frac{\delta(x)}{|x|}$.

Note that in the potential shown in figure 2(*a*), the δ-function singularity is instrumental in producing a bound state at $E_0 = 0$.

[5] At this point one may wonder whether we have lost our cherished shape invariance due to the introduction of this moderating factor. In the appendix, we show that the shape invariance indeed remains intact in the limit $\epsilon \to 0$, and so does the solvability of the model.

Naively, in the limit $\epsilon \to 0$, the potential of equation (14) appears identical to a three-dimensional oscillator with a frequency ω and angular momentum l. However, there are some more subtle but important differences. First, it is defined over the entire real axis $(-\infty < x < \infty)$ and not just the half line. For a proper communication between the two halves, we must have a 'softness' of the inverse-square term. Normalizability of the wavefunction requires that the coefficient λ of the inverse-square term be in the transition region $-\frac{1}{4} < \lambda < \frac{3}{4}$ [16]. More specifically, for $(l > 0)$, one has $0 < l(l+1) < \frac{3}{4}$ and for $(l < 0)$ one has $-\frac{1}{4} < l(l+1) < 0$. The important special case of the one-dimensional harmonic oscillator has $l = 0$: it corresponds to $l(l+1) = 0$ and no x^{-2} singularity. For transition potentials, the two independent solutions of the Schrödinger equation are both square integrable at the origin, and hence acceptable. Eigenstates for the potential $\tilde{V}_-(x, a_0)$ can be obtained from equation (6). The lowest four are

$$\begin{aligned}
&E_0 = 0 \qquad \psi_0 \propto x^{-l}\, \mathrm{e}^{-\frac{1}{4}\omega x^2} \\
&E_1 = (2l+1)\omega \qquad \psi_1 \propto x^{1+l}\, \mathrm{e}^{-\frac{1}{4}\omega x^2} \\
&E_2 = 2\omega \qquad \psi_2 \propto \left(2l - 1 + \omega x^2\right) x^{-l} \mathrm{e}^{-\frac{1}{4}\omega x^2} \\
&E_3 = 2\omega + (2l+1)\omega \qquad \psi_3 \propto \left(-2l - 3 + \omega x^2\right) x^{1+l}\, \mathrm{e}^{-\frac{1}{4}\omega x^2}.
\end{aligned} \tag{15}$$

General expressions for these eigenfunctions and corresponding eigenenergies are

$$\begin{aligned}
&E_{2n} = 2n\omega \qquad \psi_{2n} \propto x^{-l} \mathrm{e}^{-\frac{1}{4}\omega x^2} L_n^{-l-\frac{1}{2}}\left[\frac{\omega x^2}{2}\right] \\
&E_{2n+1} = 2n\omega + (2l+1)\omega \qquad \psi_{2n+1} \propto x^{1+l} \mathrm{e}^{-\frac{1}{4}\omega x^2} L_n^{l+\frac{1}{2}}\left[\frac{\omega x^2}{2}\right]
\end{aligned} \tag{16}$$

where L_n are the standard Laguerre polynomials.

In deriving the above eigenspectrum, we have implicitly made the reasonable assumption that supersymmetry is unaffected by the regularization procedure. The fact that we get finite results is an indirect confirmation of the convergence of the spectrum as $\epsilon \to 0$. Another check is that the alternative regularization method of going to the complex plane yields the same spectrum [2, 3].

3.2. New shape-invariant potential obtained from the Pöschl–Teller I potential

As a second example, we consider the Pöschl–Teller I superpotential

$$W(x, A, B) = A \tan x - B \cot x \qquad 0 < x < \pi/2. \tag{17}$$

The supersymmetric partner potentials are then given by

$$V_-(x, A, B) = A(A-1)\sec^2 x + B(B-1)\,\mathrm{cosec}^2 x - (A+B)^2 \tag{18}$$

and

$$V_+(x, A, B) = A(A+1)\sec^2 x + B(B+1)\,\mathrm{cosec}^2 x - (A+B)^2. \tag{19}$$

Here, A and B are both positive in order for $V_-(x, A, B)$ to have a zero-energy ground state.

Again, one can readily check that there are two possible relations between parameters such that above two potentials exhibit shape invariance. One of them is the conventional $(A \to A+1,\ B \to B+1)$. The second possibility is $(A \to A+1,\ B \to -B)$. As explained in the previous section, this second relationship breaks supersmmetry on the $(0, \pi/2)$ domain and it is allowed only if the domain of x is extended to range $-\pi/2 < x < \pi/2$. The first transformation among parameters $(A \to A+1,\ B \to B+1)$ has been studied extensively in the literature. It is the second transformation that yields new results and will be considered here.

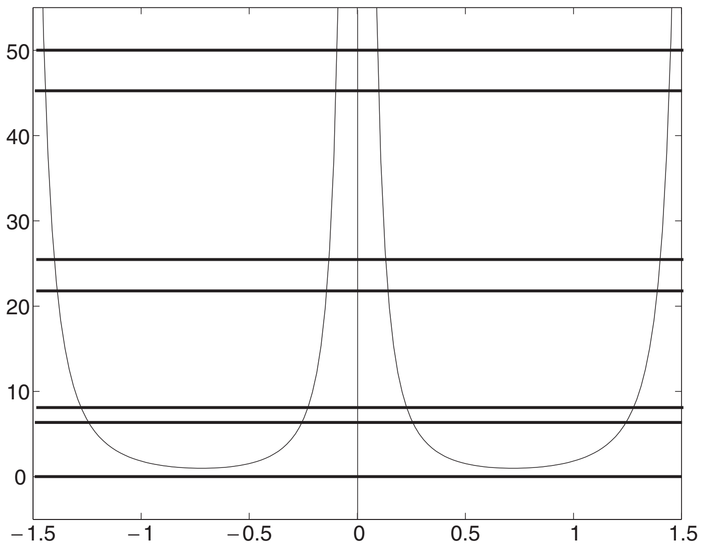

Figure 3. The potential corresponding to equation (20) for $A = 1.5$ and $B = -1/3$ and its energy spectrum.

Thus, the relationship among parameters that we consider is $(A_{k+1} = A_k + 1, \quad B_{k+1} = -B_k)$. This potential also requires a careful analysis in the vicinity of $x = 0$, where two half-axes are being sewed together. Again, the need for continuity and differentiability of the superpotential requires its regularization, as was done in equation (11) for the harmonic oscillator. A similar analysis then leads to a new singular shape-invariant potential

$$\tilde{V}_-(x, A, B) = \left[A(A-1)\sec^2 x + B(B-1)\,\mathrm{cosec}^2 x - (A+B)^2\right] + 4\,B\cot x\,\frac{x}{|x|}\,\delta(x). \tag{20}$$

This potential obeys the shape-invariance condition:

$$\tilde{V}_+(x, A, B) = \tilde{V}_-(x, A+1, -B) + (A+1-B)^2 - (A+B)^2 \tag{21}$$

and its eigenvalues and eigenfunctions are given by

$$E_0 = 0$$
$$\psi_0 \propto \cos^A x\ \sin^B x$$
$$E_1 = (A+1-B)^2 - (A+B)^2$$
$$\psi_1 \propto \cos^A x\ \sin^{-B-1} x\,[(2B-1)\cos^2 x + 1]$$
$$E_2 = (A+2+B)^2 - (A+B)^2$$
$$\psi_2 \propto \cos^A x\ \sin^B x\,[(4B+2)\cos^4 x - (6B+3)\cos^2 x + 1]$$
$$E_3 = (A+3-B)^2 - (A+B)^2$$
$$\psi_3 \propto \cos^A x\ \sin^{-B-1} x\,[(-8B^2+16B-6)\cos^6 x + (12B^2 - 32B + 13)\cos^4 x + (20B-8)\cos^2 x + 1].$$

Thus, the general formula for eigenvalues is

$$E_n = (A + n + (-1)^n B)^2 - (A+B)^2. \tag{22}$$

The eigenspectrum is shown in figure 3, whereas the superpotential and potential corresponding to two choices of A, B are shown in figure 4.

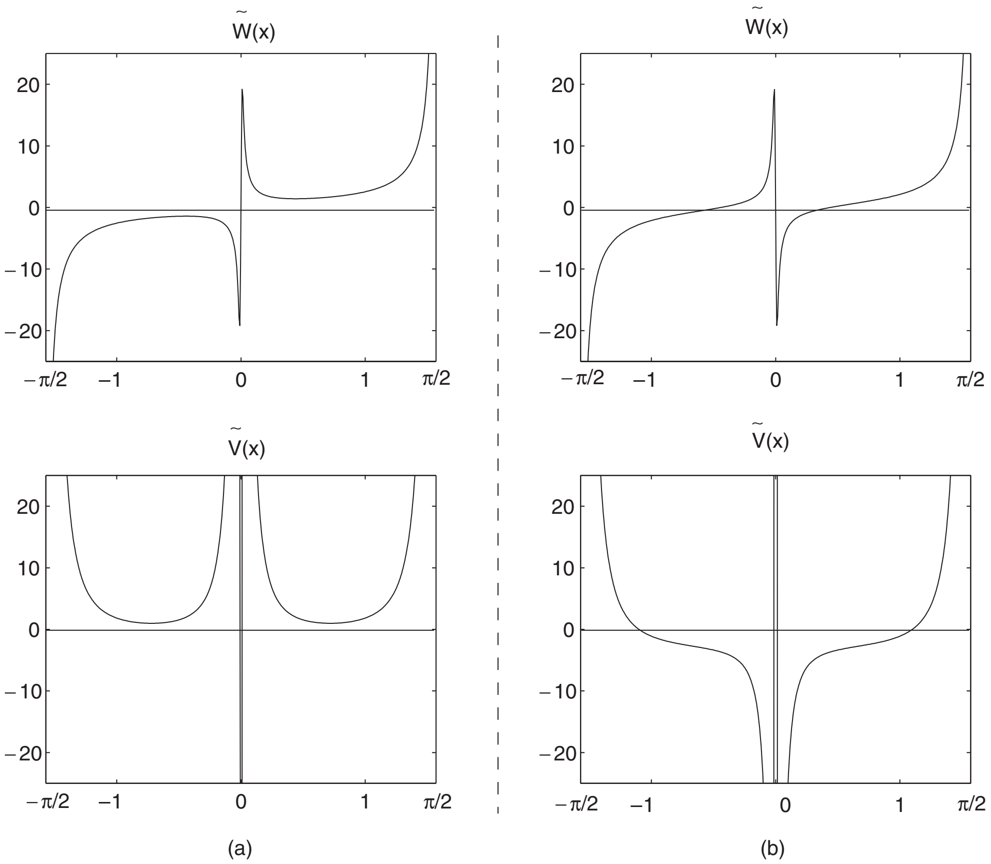

Figure 4. The superpotential $\tilde{W}$ and the corresponding potential (equation (20)) for $A = 1.5$ for the two cases (*a*) $B = -1/3$ and (*b*) $B = 1/3$.

Note that, to avoid level crossing, we must have $E_n > E_{n-1}$. This leads to the constraint $-\frac{1}{2} < B < \frac{1}{2}$. Interestingly, this is the same constraint that one needs for the normalizability of the wavefunction at the origin and hence to the possibility of communication between regions $(-\frac{1}{2}\pi, 0)$ and $(0, \frac{1}{2}\pi)$ of the domain.

3.3. New shape-invariant potential obtained from the Pöschl–Teller II potential

The last example that we consider is that of the Pöschl–Teller II potential described by

$$W(x, A, B) = A \tanh x - B \coth x \qquad 0 < x < \infty. \tag{23}$$

Here, A and B both need to be positive and satisfy the condition $A > B$ for the potential $V_-(x, A, B)$ to have a zero-energy ground state and to ensure unbroken supersymmetry. The supersymmetric partner potentials are then given by

$$V_+(x, A, B) = -A(A-1)\,\mathrm{sech}^2 x + B(B+1)\,\mathrm{cosec}^2 x + (A-B)^2 \tag{24}$$

and

$$V_-(x, A, B) = -A(A+1)\sec^2 x + B(B-1)\,\mathrm{cosec}^2 x + (A-B)^2. \tag{25}$$

Here too we have two possible relations between parameters for these potentials to be shape invariant. They are $(A \to A-1,\ \ B \to B+1)$, and $(A \to A-1,\ \ B \to -B)$. As explained before in the last two examples, the second transformation requires an extension of the range to $(-\infty, \infty)$. The new singular potential generated for this case is given by

$$\tilde{V}_{-}(x, A, B) = \big[-A(A-1)\,\mathrm{sech}^2 x + B(B+1)\,\mathrm{cosech}^2 x + (A-B)^2 \big] \\ +4\,B \coth x\, \frac{x}{|x|}\, \delta(x). \qquad (26)$$

The shape-invariance condition obeyed by this potential is given by

$$\tilde{V}_{+}(x, A, B) = \tilde{V}_{-}(x, A+1, -B) + (A-B)^2 - (A-1+B)^2 \qquad (27)$$

and the eigenvalues and eigenfunctions are given by

$$\begin{aligned}
&E_0 = 0\\
&\psi_0 \propto \cosh^{-A} x\ \ \sinh^{B} x\\
&E_1 = (A-B)^2 - (A-1+B)^2\\
&\psi_1 \propto \cosh^{-A} x\ \ \sinh^{-B-1} x\ [(2B-1)\cosh^2 x + 1]\\
&E_2 = (A-B)^2 - (A-2-B)^2\\
&\psi_2 \propto \cosh^{A} x\ \ \sinh^{B} x\ [(4B+2)\cosh^4 x - (6B+3)\cosh^2 x + 1]\\
&E_3 = (A-B)^2 - (A-3+B)^2\\
&\psi_3 \propto \cosh^{A} x\ \ \sinh^{-B-1} x\ \big[(8B^2 - 16B + 6)\cosh^6 x - (12B^2 - 32B + 13)\cosh^4 x\\
&\qquad\qquad -(20B-8)\cosh^2 x - 1\big].
\end{aligned} \qquad (28)$$

Thus, the general formula for eigenvalues is

$$E_n = (A-B)^2 - (A - n - (-1)^n B)^2. \qquad (29)$$

Again, to steer clear of the level crossing problem, we must have $E_n > E_{n-1}$. This leads to the constraint $-\frac{1}{2} < B < \frac{1}{2}$, which, as stated earlier, is the same constraint that one needs for the normalizability of the wavefunction at the origin and for an effective communication between two halves of the x-axis.

4. Potential algebra

So far, we have discussed three types of new solvable singular potential. We will now derive the potential algebra underlying them. We will show that the algebra based on the generators $\{J_+, J_-, J_3\}$ is nonlinear [6–10]. Potential algebras provide an alternative way of getting the eigenvalues by algebraic means.

Consider the following ansatz:

$$J_+ = c^{-1}\,\mathcal{A}^{\dagger}\,(x, \alpha(N), \beta(N)) \qquad J_- = \mathcal{A}\,(x, \alpha(N), \beta(N))\,c \qquad J_3 = N \equiv c^{\dagger} c \qquad (30)$$

where c, $c^{\dagger}$ and c^{-1} are three operators satisfying $[c, c^{\dagger}] = 1$, and $c\,c^{-1} = c^{-1}\,c = 1$. An example of such operators is given by $c = \mathrm{e}^{\mathrm{i}\phi}$, $c^{-1} = \mathrm{e}^{-\mathrm{i}\phi}$ and $c^{\dagger} = \mathrm{i}\partial_{\phi}\,\mathrm{e}^{-\mathrm{i}\phi}$, where ϕ is some arbitrary real variable. The operators $\mathcal{A}$ and $\mathcal{A}^{\dagger}$ of equation (30) are obtained from equation (2) via the substitution $a_0 \equiv \{A, B\} \to \{\alpha(N), \beta(N)\}$, where α and β are real, arbitrary functions to be determined later. We can readily check that

$$[J_3, J_{\pm}] = \pm J_{\pm}\ ,\ [J_+, J_-] = -R(J_3) \equiv g(\alpha(N), \beta(N)) - g(\alpha(N-1), \beta(N-1)). \qquad (31)$$

The last commutation relation is a consequence of the algebraic shape-invariance condition [9]

$$H_+(x, \alpha(N), \beta(N)) - H_-\,(x, \alpha(N-1), \beta(N-1)) \\ = g(\alpha(N-1), \beta(N-1)) - g(\alpha(N), \beta(N)) \qquad (32)$$

which is the operatorial 'twin' of the classical shape-invariance condition equation (4) obtained via the mappings $\{A_0, B_0\} \rightarrow \{\alpha(N), \beta(N)\}$ and $\{A_1, B_1\} \rightarrow \{\alpha(N-1), \beta(N-1)\}$ respectively.

The functions $\alpha(N)$ and $\beta(N)$ are determined by requiring that the change $\alpha(N) \rightarrow \alpha(N-1)$ and $\beta(N) \rightarrow \beta(N-1)$ correspond to the change of parameters $a_0 \rightarrow a_1$. For example, $\alpha(N) = A - N$ corresponds to a translational change of parameters $A_0 \rightarrow A_1 = A_0 + 1$, because $\alpha(N-1) = \alpha(N)+1$. Similarly, $\beta(N) = (-1)^N$ corresponds to the reflection $B_0 \rightarrow B_1 = -B_0$, since $\beta(N-1) = -\beta(N)$.

For any shape-invariant potential, we know the function $g(\alpha, \beta)$, which explicitly gives the potential algebra (31). From its representations, we can obtain the energy spectrum for the given problem.

To find a representation of the potential algebra, let us consider a set of eigenvectors common to both $H_- = J_+J_-$ and $J_3 = N$ denoted by $\{|n\rangle, n = 0, 1, \ldots\}$. The action of J_+, J_- and J_3 on this basis is given by

$$J_+|n\rangle = a(n+1)|n+1\rangle \qquad J_-|n\rangle = a(n)|n-1\rangle \qquad J_3|n\rangle = n|n\rangle. \tag{33}$$

Here we have chosen, without any loss of generality, the coefficients $a(n)$ to be real. Note that since $J_-|0\rangle = 0$, we have the initial condition $a(0) = 0$. There is a connection between the coefficients $a(n)$ and the eigenspectrum of the Hamiltonian. Observe that

$$H_-(x, \alpha(N-1), \beta(N-1))\,|n\rangle = J_+J_-|n\rangle = a^2(n)|n\rangle. \tag{34}$$

Therefore, in order to find the spectrum of the Hamiltonian we have to determine the coefficients $a^2(n)$. This can be done by projecting the last equation from (31) on $|n\rangle$ and solving the resulting equation recursively. Thus, we obtain $a^2(n) - a^2(n+1) = g(n) - g(n-1)$ having the solution $a^2(n) = g(-1) - g(n-1)$. Here we have denoted $g(n) \equiv g(\alpha(n), \beta(n))$. But $a^2(n)$ corresponds to the eigenvalues of the Hamiltonian $H_-(x, \alpha(N-1), \beta(N-1))$, or 'classically' speaking to the shifted set parameters a_1. Therefore the eigenenergies of the initial Hamiltonian $H_-(x, \alpha(N), \beta(N))$ (corresponding to the set of parameters a_0) are

$$E_n = g(\alpha(0), \beta(0)) - g(\alpha(n), \beta(n)). \tag{35}$$

We make contact with equation (6) by observing that $\{\alpha(n-k), \beta(n-k)\} \equiv a_k$.

4.1. The harmonic oscillator

To show how our procedure works, it is instructive to build explicitly the potential algebra of the harmonic oscillator. The superpartner potentials V_- and V_+ are given in equations (7) and (9) respectively. Under the change of parameters $\{l, \omega\} \rightarrow \{l-1, \omega\}$ we have the following shape-invariance condition:

$$H_+(x, l, \omega) = H_-(x, l-1, \omega) + (-2\omega\,(l-1)) - (-2\omega\, l).$$

To build the potential algebra, first we find the functions α and β associated with the above change of parameters. We have immediately $\alpha(N) = l + N$, $\beta(N) = \omega$. Next, we can build the concrete realization of the potential algebra using the ansatz (30) and the superpotential from (8). The resulting generators

$$\begin{aligned} J_+ &= c^{-1}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\,x\,\omega + \frac{l+N}{x}\right) \\ J_- &= \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\,x\,\omega + \frac{l+N}{x}\right) c \\ J_3 &= N \equiv c^{\dagger}c \end{aligned} \tag{36}$$

satisfy the 'canonical' commutation relations (31), where the function g is given by $g(N) \equiv g(\alpha(N), \beta(N)) = -2\omega\,(l + N)$. Finally, using the formula (35) we get the spectrum $E_n = g(0) - g(n) = 2\omega\, n$, which is exactly what we have expected.

Next, let us consider the new singular shape-invariant potential corresponding to the change of parameters $\{l, \omega\} \to \{-l, \omega\}$. In this case $\alpha(N) = -(-1)^N l$ and $\beta(N) = \omega$. From equation (30) we get

$$J_+ = c^{-1}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\,x\,\omega - \frac{(-1)^N l}{x}\right) \qquad J_- = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\,x\,\omega - \frac{(-1)^N l}{x}\right) c$$
$$J_3 = N \equiv c^\dagger c. \tag{37}$$

The commutation relations (31) together with the algebraic shape-invariance condition (32) yield in this case $[J_+, J_-] = -\omega\,(-2(-1)^N l + 1)$ from where we get $g(N) = \omega\,((-1)^N l - N)$. Therefore, the resulting eigenspectrum (35) is $E_n = \omega\, n + \omega\, l\,(1 - (-1)^n)$.

4.2. The Pöschl–Teller I potential

We build the algebraic model for the new shape-invariant Pöschl–Teller-I-like potential by taking into account that corresponding to the change of parameters $\{A, B\} \to \{A + 1, -B\}$ we have $\alpha(N) = A - N$ and $\beta(N) = (-1)^N B$. Then, using the superpotential (17) one gets the following expressions for the generators of the associated potential algebra:

$$\begin{aligned} J_+ &= c^{-1}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + (A - N)\tan x - (-1)^N B\cot x\right) \\ J_- &= \left(-\frac{\mathrm{d}}{\mathrm{d}x} + (A - N)\tan x - (-1)^N B\cot x\right) c \qquad J_3 = N \equiv c^\dagger c. \end{aligned} \tag{38}$$

Using as before the algebraic shape-invariance condition (32) we obtain in this case $[J_+, J_-] = -(A+N+1+(-1)^{(N+1)}B)^2+(A+N+(-1)^N B)^2$. Therefore we get $g(N) = -(A+N+(-1)^N B)^2$ and the corresponding eigenspectrum $E_n = g(0) - g(n) = -(A + B)^2 + (A - n + (-1)^n B)$.

4.3. The Pöschl–Teller II potential.

For the new Pöschl–Teller-II-potential-like case, for the change of parameters $\{A, B\} \to \{A - 1, -B\}$ we have $\alpha(N) = -(A - N)$ and $\beta(N) = (-1)^N B$ and the corresponding algebra is therefore generated by

$$\begin{aligned} J_+ &= c^{-1}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + (-A + N)\tanh x - (-1)^N B\coth x\right) \\ J_- &= \left(-\frac{\mathrm{d}}{\mathrm{d}x} + (-A + N)\tanh x - (-1)^N B\coth x\right) c \qquad J_3 = N \equiv c^\dagger c. \end{aligned} \tag{39}$$

In the above representation the explicit form of the superpotential (23) was taken into account. The commutation relations (31) together with the algebraic shape-invariance condition (32) yield in this case $g(N) = (-A + N - (-1)^N B)^2$. Using (35), one obtains as expected, the eigenspectrum for this potential $E_n = g(0) - g(n) = (A + B)^2 - (-A + n - (-1)^n B)^2$.

5. Conclusions and comments

We have generated several new shape-invariant potentials on the whole line starting from well known potentials on the half line. To ensure continuity and differentiability of the superpotential, our procedure requires a regularization at the origin. This extension not only maintains shape invariance, it also allows the possibility of a new transformation among parameters ($B \to -B$) that was not allowed on the half-axis. This transformation results in new superpotentials, albeit singular, that are defined over the entire real axis and have richer spectra than those defined over the half-axis. It is shown further that the eigenspectra of these new real singular shape-invariant potentials may also be derived from a nonlinear potential algebra.

Since we have obtained and discussed the exact eigenvalues and eigenfunctions of three new singular potentials using the machinery of SUSYQM, it is of interest to ask what one gets in the WKB approximation. Let us recall that Comtet *et al* have shown the exactness of the SWKB quantization condition [13, 17]

$$\int_{x_1}^{x_2} \sqrt{E_n - W^2}\,\mathrm{d}x = n\pi\hbar$$

for all known shape-invariant problems with unbroken SUSY where parameters are related by $a_1 = a_0 + \delta$ [20].

For broken SUSY, Inomata and Junker [18] gave the quantization condition

$$\int_{x_1}^{x_2} \sqrt{E_n - W^2}\,\mathrm{d}x = [n + 1/2]\pi\hbar.$$

For both cases, the turning points x_1, x_2 are solutions of $W^2(x) = E_n$.

Our new singular potentials allow a change of parameters that, if considered in half-axes only, leads the system to alternate between unbroken and broken phases of supersymmetry as $a_k \to a_{k+1}$ [20]. It is interesting to note that the spectrum of these singular potentials can be derived, using a somewhat more complex but exact quantization condition which alternates between the broken and unbroken SUSY cases:

$$\int_{x_1}^{x_2} \sqrt{E_n - W^2}\,\mathrm{d}x = [n + 1/2P_n]$$

where P_n is equal to $[1 - (-1)^n]/2$.

Acknowledgments

Partial financial support from the US Department of Energy is gratefully acknowledged. RD would like to thank the Department of Atomic Energy, Government of India for a research grant, and the Physics Department of the University of Illinois at Chicago for warm hospitality.

Appendix

In this appendix, we show that the shape invariance of our new potentials is maintained during the process of extending the domain to the whole real axis and introducing the moderating factor $f(x, \epsilon)$. Let us recall that our old superpotential $W(x, A, B)$ is of the form $A\ \Phi(x) + B\ [\Phi(x)]^{-1}$, where the function $\Phi(x)$ is x, $\tan x$ or $\tanh x$ for the harmonic oscillator, Pöschl–Teller I and Pöschl–Teller II respectively[6].

[6] The change of parameters associated with shape invariance in these three potentials are of the form

$$A \longrightarrow \tilde{A} = \begin{cases} A \\ A + 1 \\ A - 1. \end{cases} \qquad \text{and} \qquad B \longrightarrow \tilde{B} = -B.$$

Note that in all cases, $[\Phi(x)]^{-1} \longrightarrow 1/x$ at the origin. $W(x, A, B)$ is replaced by a regularized, continuous superpotential $\tilde{W}(x, A, B, \epsilon)$ given by

$$\tilde{W}(x, A, B, \epsilon) = W(x, A, B)\ f(x, \epsilon) \tag{A1}$$

where $f(x, \epsilon)$ is unity everywhere except in a small region of order ϵ around $x = 0$. One such function $f(x, \epsilon)$ is given by $\tanh^2(x/\epsilon)$. In the limit $\epsilon \to 0$, we assume that the $f(x, \epsilon) \to 1$ and $\frac{\mathrm{d}f(x,\epsilon)}{\mathrm{d}x} \to 2\frac{x}{|x|}\delta(x)$. The potentials $\tilde{V}_{\mp}(x, A, B, \epsilon)$ corresponding to the superpotential $\tilde{W}(x, A, B, \epsilon)$ are then given by

$$\tilde{V}_{\mp}(x, A, B, \epsilon) = W^2(x, A, B) f^2(x, \epsilon) \mp \left(\frac{\mathrm{d}W(x, A, B)}{\mathrm{d}x}\ f(x, \epsilon) + \frac{\mathrm{d}f(x, \epsilon)}{\mathrm{d}x}\ W(x, A, B) \right). \tag{A2}$$

Now

$$\begin{aligned} \tilde{V}_{+}(x, A, B, \epsilon) - \tilde{V}_{-}(x, \tilde{A}, \tilde{B}, \epsilon) &= V_{+}(x, A, B) - V_{-}(x, \tilde{A}, \tilde{B}) \\ &\quad + \frac{\mathrm{d}f(x, \epsilon)}{\mathrm{d}x} \left(W(x, A, B) + W(x, \tilde{A}, \tilde{B}) \right) \\ &= R(A, B) + \frac{\mathrm{d}f(x, \epsilon)}{\mathrm{d}x} \left(B + \tilde{B} \right) [\Phi(x)]^{-1} \\ &= R(A, B) \end{aligned} \tag{A3}$$

where we have used the limits of f and f' and $\tilde{B} = -B$. This establishes the shape invariance of the regularized superpotential.

References

[1] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **24** 5243
Cannata F, Junker G and Trost J 1998 *Phys. Lett.* A **246** 219
Bender C M, Boettcher S and Meisinger P N 1999 *J. Math. Phys.* **40** 2201
Andrianov A A, Ioffe M V, Cannata F and Dedonder J P 1999 *Int. J. Mod. Phys.* A **14** 2675
Znojil M 1999 *Phys. Lett.* A **264** 108
Znojil M 1999 *J. Phys. A: Math. Gen.* **32** 4563
Bagchi B and Quesne C 2000 *Phys. Lett.* A **273** 285

[2] Znojil M 1999 *Phys. Lett.* A **259** 220

[3] Znojil M 1999 *Preprint* arXiv: quant-ph/9911116

[4] Gangopadhyaya A and Sukhatme U 1996 *Phys. Lett.* A **224** 5

[5] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
Gendenshtein L E 1983 *Pis. Eksp. Teor. Fiz.* **38** 299 (Engl. Transl. 1983 *JETP Lett.* **38** 356)
Gendenshtein L E and Krive I V 1985 *Sov. Phys.–Usp.* **28** 645

[6] Wu J and Alhassid Y 1990 *Phys. Rev.* A **31** 557
Cheung H Y 1988 *Nuovo Cimento* B **101** 193

[7] Wu J, Alhassid Y and Gürsey F 1989 *Ann. Phys., NY* **196** 163

[8] Gangopadhyaya A, Mallow J V and Sukhatme U P 1997 *Proc. Workshop on Supersymmetric Quantum Mechanics and Integrable Models* ed Henrik Aratyn *et al* (Berlin: Springer)

[9] Gangopadhyaya A, Mallow J V and Sukhatme U P 1998 *Phys. Rev.* A **58** 4287
Chaturvedi S, Dutt R, Gangopadhyaya A, Panigrahi P, Rasinariu C and Sukhatme U 1998 *Phys. Lett.* A **248** 109
Dutt R, Gangopadhyaya A, Rasinariu C and Sukhatme U 1999 *Phys. Rev.* A **60** 3482

[10] Balantekin A B 1998 *Phys. Rev.* A **57** 4188
(Balantekin A B 1997 *Preprint* quant-ph/9712018)

[11] Witten E 1981 *Nucl. Phys.* B **188** 513
Cooper F and Freedman B 1983 *Ann. Phys., NY* **146** 262
Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917

[12] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 268 and references therein

[13] Dutt R, Khare A and Sukhatme U 1988 *Am. J. Phys.* **56** 163

[14] Dutt R, Khare A and Sukhatme U 1986 *Phys. Lett.* B **181** 295

[15] Barclay D, Dutt R, Gangopadhyaya A, Khare A, Pagnamenta A and Sukhatme U 1993 *Phys. Rev.* A **48** 2786

[16] Gangopadhyaya A, Panigrahi P K and Sukhatme U P 1994 *J. Phys. A: Math. Gen.* **27** 4295
Landau L D and Lifshitz E M 1977 *Quantum Mechanics* (Oxford: Pergamon)
Frank W M, Land D J and Spector R M 1971 *Rev. Mod. Phys.* **43** 36

[17] Comtet A, Bandrauk A D and Campbell D K 1985 *Phys. Lett.* B **150** 159

[18] Inomata A and Junker G 1991 *Proc. Adriatic Research Conf. on Path-Integration and its Applications* vol 3–6 *(Trieste, ITCTP)*

[19] Dutt R, Gangopadhyaya A, Khare A, Pagnamenta A and Sukhatme U 1993 *Phys. Lett.* A **174** 363

[20] Barclay D T, Khare A and Sukhatme U 1993 *Phys. Lett.* A **183** 263